Second order differential equations

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This handout is meant to give you a couple more example of all the techniques discussed in chapter 4, to counterbalance all the dry theory and complicated applications in the differential equations book! Enjoy! :)

1 Homogeneous equations

1.1 Distinct roots: Solve $y'' - 5y' + 6y = 0$

The auxiliary polynomial is $r^2 - 5r + 6 = (r-2)(r-3)$, which has zeros $r = 2, 3$.

$$y(t) = Ae^{2t} + Be^{3t}$$

1.2 Complex roots: Solve $y'' - 8y' + 25y = 0$

Aux: $r^2 - 8r + 25 = 0$, which gives $r = \frac{8 \pm \sqrt{-36}}{2} = 4 \pm 3i$ (use the quadratic formula).

$$y(t) = Ae^{4t}cos(3t) + Be^{4t}sin(3t)$$

1.3 Repeated roots: Solve $y'' - 4y' + 4y = 0$

Aux: $r^2 - 4r + 4 = (r-2)^2 = 0$, which gives $r = 2$, a root with multiplicity 2. Hence:

$$y(t) = Ae^{2t} + Bte^{2t}$$

2 Undetermined coefficients

2.1 Solve $y'' - 5y' + 6y = e^t$

The general solution is $y(t) = y_0(t) + y_p(t)$, where:

- $y_0(t)$ is the general solution to $y'' - 5y' + 6y = 0$
- $y_p(t)$ is *one particular* solution to $y'' - 5y' + 6y = e^t$

On the previous page, we found $y_0(t) = Ae^{2t} + Be^{3t}$

To find y_p, in this example guess y_p has the form $y_p(t) = Ae^t$, and plug this back into the original equation, you get:

$$(Ae^t)'' - 5(Ae^t)' + 6(Ae^t) = e^t$$

$$2Ae^t = e^t$$

$$2A = 1$$

Hence $A = \frac{1}{2}$, and $y_p(t) = \frac{1}{2}e^t$.

So the general solution to $y'' - 5y' + 6y = e^t$ is:

$$y(t) = y_0(t) + y_p(t) = Ae^{2t} + Be^{3t} + \frac{1}{2}e^t$$

2.2 How to guess the form of y_p

The main idea is: Always try the most complicated case you can think of!

2.2.1 $y'' - 2y' + y = t^3 e^{2t}\cos(4t)$

Aux: $r^2 - 2r + 1 = (r-1)^2 = 0$, so $r = 1$.

Here you have to guess:

$$y_p(t) = (At^3 + Bt^2 + Ct + D)e^{2t}\cos(4t) + (A't^3 + B't^2 + C't + D')e^{2t}\sin(4t)$$

Note that we have a double root, but here it doesn't matter because the right hand side doesn't involve e^t

2.2.2 $y'' - 3y' + 2y = t^2e^t$

Aux: $r^2 - 3r + 2 = (r-1)(r-2) = 0$, which gives $r = 1, 2$.

Here, one of the roots *coincides* with the right-hand-side, so we have to add an extra t in our solution:

$$y_p(t) = t(At^2 + Bt + C)e^t$$

Note: If we had e^{4t} instead of e^t, we wouldn't need to add that t.

2.2.3 $y'' - 2y' + y = e^t$

Aux: $r^2 - 2r + 1 = (r-1)^2 = 0$, which gives $r = 1$

Here we really have the worst-case scenario: A double root which coincides with the right hand side, so we have to add a factor of t^2 in our solution!

$$y_p(t) = At^2e^t$$

2.2.4 $y'' - 8y' + 25y = e^{4t}\cos(3t)$

Aux: The roots are $r = 4 \pm 3i$ (see above)

The root $4+3i$ coincides with the $e^{4t}\cos(3t)$ factor, so we have to add an extra factor of t:

$$y_p(t) = Ate^{4t}\cos(3t) + Bte^{4t}\sin(3t)$$

That t factor wouldn't be present if we had $e^{5t}\cos(3t)$ on the right-hand-side, or even just $\cos(3t)$.

3 Superposition principle

3.1 Solve $y'' + y = e^t + \cos(t) + t$

This just says that to find a particular solution y_p to this equation, just find particular solutions y_1, y_2, y_3 to the equations:

$$y'' + y = e^t$$

$$y'' + y = \cos(t)$$

$$y'' + y = t$$

And add them together to get $y_p(t) = y_1(t) + y_2(t) + y_3(t)$

4 Variation of parameters

4.1 Solve $y'' + y = \tan^2(t)$

First of all, make sure that the coefficient of y'' is 1. (for example, if the equation was $2y'' + 2y = 2\tan^2(t)$, we would divide the equation by 2)

Now $y(t) = y_0(t) + y_p(t)$ as usual, where $y_0(t) = A\cos(t) + B\sin(t)$ (the roots of the aux polynomial are $r = \pm i$)

To find y_p, use the method of <u>Variation of parameters</u>

Suppose $y_p(t) = v_1(t)\cos(t) + v_2(t)\sin(t)$ (where v_1 and v_2 are functions)

Let $f(t) = \cos(t)$ and $g(t) = \sin(t)$ (the solutions to the hom. equation), and let:

$$\widetilde{W}(t) = \begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Now solve:

$$\widetilde{W}(t)\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \tan^2(t) \end{bmatrix}$$

Note: The first row of the RHS is always 0, and the second row is always the inhomogeneous term.

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = (\widetilde{W}(t))^{-1}\begin{bmatrix} 0 \\ \tan^2(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} 0 \\ \tan^2(t) \end{bmatrix} = \begin{bmatrix} -\sin(t)\tan^2(t) \\ \cos(t)\tan^2(t) \end{bmatrix}$$

Where I have used the formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^1 = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence, we get:

$$v_1'(t) = -\sin(t)\tan^2(t) = -\frac{\sin^3(t)}{\cos^2(t)}$$

$$v_2'(t) = \cos(t)\tan^2(t) = \frac{\sin^2(t)}{\cos(t)}$$

So:

$$v_1(t) = \int -\frac{\sin^3(t)}{\cos^2(t)}dt = -\cos(t) - \sec(t)$$

$$v_2(t) = \int \frac{\sin^2(t)}{\cos(t)}dt = \ln|\sec(t)+\tan(t)| - \sin(t)$$

Note: It's OK to use your calculator or Wolfram Alpha to evaluate those integrals!

Hence, putting everything together, we get:

$$y_p(t) = v_1(t)\cos(t) + v_2(t)\sin(t) = (-\cos(t)-\sec(t))\cos(t) + (\ln|\sec(t)+\tan(t)| - \sin(t))\sin(t)$$

$$y_p(t) = -2 + \ln|\sec(t)+\tan(t)|\sin(t)$$

Whence:

$$y(t) = y_0(t) + y_p(t) = A\cos(t) + B\sin(t) + -2 + \ln|\sec(t) + \tan(t)|\sin(t)$$